Ascent or Descent from Satellite Orbit by Low Thrust

YUN Y. SHI* AND MARTIN C. ECKSTEIN†
Douglas Aircraft Company Inc., Santa Monica, Calif.

The problem of ascent or descent from an initially Keplerian orbit by a constant low thrust is investigated by the two-variable expansion procedure. The slowly varying elements of the Keplerian orbit are explicitly computed, and a simple integral relating the eccentricity and semimajor axis is derived and used to justify the omission of higher order terms in the eccentricity for ascent from almost circular initial orbits. The corresponding assumption for descent from an almost circular orbit is shown to be inconsistent. The present higher order theory that is valid for arbitrary initial eccentricities is shown to contain earlier results for circular starting orbits and gives simple explicit formulas for the solution at small eccentricities. The present results do, however, suffer from the deficiency of earlier solutions in that the validity is restricted to periods before escape velocity is reached for the case of ascending orbits.

1. Introduction

THE problem of ascent from circular orbit by a small thrust in either the radial or circumferential directions was first investigated by Tsien.[1] Tsien gave an exact solution for the case of radial thrust and developed an approximate first-order solution for the circumferential thrust case.

Recently, Ting and Brofman[2] extended Tsien's solution to one higher order and allowed the thrust to be arbitrarily oriented while remaining constant. Their approach consists of splitting the dependence of the radius into oscillatory and non-oscillatory parts and using an expansion procedure combining the features of singular perturbations and the method of averaging. More recently, Nayfeh[3] also developed a solution for the case treated by Ting and Brofman, using a more systematic asymptotic method. The quoted results were all concerned with the case of circular initial orbits and do not shed any light on the more interesting case of an arbitrary elliptic starting orbit.

In the closely related problem of ascent by tangential thrust, Zee[4] and Cohen[5] attempted to solve for the oscillatory terms in the radius contributed by the ellipticity of the initial orbit. Their solutions suffer from the presence of unnecessary assumptions that restrict the validity of the results to small eccentricity in spite of the complicated approaches proposed.

In another recent paper, King[6] considered the question of descent from a parking orbit by low thrust, but did not discuss the analytical implications of this reversal of the usual role of low thrust.

In this paper, the following generalizations are made. The initial elliptic orbit is not restricted in any way, and the implications of negative thrust (i.e., thrust opposing the motion) are fully investigated. The solution is developed to second order by the two-variable expansion procedure developed by Cole and Kevorkian[7] and Kevorkian.[8] This method, which has been applied to two other problems in celestial mechanics by Eckstein, Shi and Kevorkian,[9,10] is a generalized asymptotic expansion procedure that provides uniformly valid developments for the coordinates appropriate to satellite problems.

At this point, a brief review of the diverse methods applied to the solution of this problem is appropriate. In the discussion of takeoff from a circular orbit by tangential thrust, Tsien[1] encountered a problem in matched asymptotic expansion (cf. Kaplun and Lagerstom[11] for a discussion of this subject) to first order, simply because the oscillatory terms multiplied by the eccentricity were absent for this degenerate case. It is in the pursuit of the preceding line of attack that complicated splitting of oscillatory and non-oscillatory terms was needed in Ting and Brofman's[2] approach. Kevorkian[8] has shown that, for problems consisting of slowly modulated oscillations such as the present case, one cannot represent the solution in terms of one variable,‡ and this is the motivation for the development of a procedure involving the explicit use of more than one independent variable. It should also be pointed out that the equivalence of the aforementioned with other generalized asymptotic expansion procedures (cf. Morrison[12] for a comparison of the modified method of averaging with the two-variable method) leaves open the choice of diverse methods which are indeed equivalent under a careful scrutiny.

A case in point is the almost exact correspondence of Nayfeh's[3] derivative expansion method and the two-variable method, or the fact that the stroboscopic method employed by Zee[4] (cf. Minorsky[13] for a brief discussion) is a crude version of the method of averaging.

In addition to this diversity of equivalent methods for the asymptotic solution of problems in nonlinear oscillations, occasionally direct solutions are also attempted (cf. Cohen[5]). This approach starts by neglecting a sufficient number of terms (by a priori and sometimes inconsistent estimates) to allow the emergence of a manipulative solution. One also encounters perturbation solutions (cf. Johnson and Stumpf[14]) containing mixed secular terms and thus valid only for periods of order unity.

The validity of the present results is evaluated for both ascending and descending trajectories, showing the greater generality of the latter and pointing out the failure of all attempts made so far in developing uniformly valid ascending orbits up to and beyond escape.

Since the present problem is concerned with constant thrust, it may be worth mentioning that extensive numerical studies have been discussed by Irving.[15] Irving has indicated that, for almost all the optimum trajectories run, the absolute magnitude of the thrust acceleration stays nearly constant throughout the run; it is not too surprising, then, that a constant acceleration caused by thrust appears also to be desirable for the escape problem. He also shows that the

Received March 17, 1966; revision received July 18, 1966. This work was sponsored by the Douglas Aircraft Company, under the independent Research and Development Fund 81301-016 42147. The authors wish to express their appreciation to J. Kevorkian for his many valuable suggestions and comments during the course of this study.

* Senior Engineer/Scientist. Member AIAA.
† Engineer/Scientist Specialist. Member AIAA.

‡ Incidentally, the failure of the so-called PLK method, involving the expansion in terms of one stretched variable, as pointed out by Nayfeh[3] is also evident without detail demonstration for the problem.

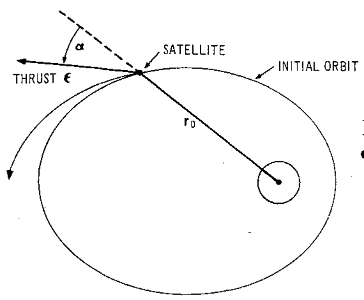

Fig. 1 Ascent from an elliptic orbit by a small thrust.

solution caused by the constant tangential or the constant circumferential thrust is nearly optimum up to escape.

2. Equations of Motion

The planar motion of a satellite accelerated by a low thrust in a central force field is governed by the following equations:

$$(d^2r/dt^2) - r(d\theta/dt)^2 = -(1/r^2) + \epsilon \cos\alpha \quad (1a)$$

$$(d/dt)[r^2(d\theta/dt)] = r\epsilon \sin\alpha \quad (1b)$$

where r and θ are polar coordinates in the plane of motion. The small dimensionless parameter ϵ denotes the ratio of the magnitude of the thrust vector to the initial weight of the satellite at its initial distance. The orientation of the thrust vector is assumed to be in the plane of motion and is specified by the angle α measured from the local vertical towards the forward thrust direction; both ϵ and α are assumed to be constant. The small parameter ϵ is always positive, and $\sin\alpha$ is positive or negative depending on the direction of the thrust (see Fig. 1). The radius vector r and the time t are nondimensionalized, such that the unit of length is the initial value of the radius vector r_0 and the unit of time is $(r_0/g)^{1/2}$, where g is the gravitational acceleration at $t = 0$.

The present formulation can be reduced to a form convenient for problems of this type by introducing the following set of new variables

$$u = 1/r \quad (2a)$$

$$f = [r^2(d\theta/dt)]^{-1} \quad (2b)$$

and regarding θ as the independent variable. The governing equations (1a) and (1b) then become

$$(d^2u/d\theta^2) + u = f^2 - \epsilon[(f^2/u^2)\cos\alpha + (f^2/u^3)(du/d\theta)\sin\alpha] \quad (3a)$$

$$(df/d\theta) = -\epsilon(f^3/u^3)\sin\alpha \quad (3b)$$

The foregoing system of nonlinear equations for u and f is to be solved in terms of θ, subject to the following initial conditions at $\theta = 0$,

$$f(0) = f_{00} \quad (4a)$$

$$u(0) = f_{00}^2[1 + e_0 \cos\omega_0] = 1 \quad (4b)$$

$$du(0)/d\theta = f_{00}^2 e_0 \sin\omega_0 \quad (4c)$$

where f_{00} and ω_0 are arbitrary constants and $0 \leq e_0 < 1$. The significance of expressing the initial conditions in the foregoing form will be apparent later. Quite generally however, e_0 and ω_0 can be expressed in terms of the initial values of f, u, and $du/d\theta$ according to equations (4a-4c) as follows:

$$e_0^2 = 1/f(0)^4\{[1 - f(0)^2] + [du(0)/d\theta]^2\} \quad (5a)$$

$$\omega_0 = \tan^{-1}\left[\frac{du(0)/d\theta}{1 - f(0)^2}\right] \quad (5b)$$

3. Expansion Procedure

The two-variable expansion procedure, developed by Cole and Kevorkian,[7] is particularly suited to the development of a uniformly valid asymptotic representation for the solution of equations such as (3a) and (3b). The reader is referred to the work of Kevorkian[8] for a detailed discussion of this method. In addition to the foregoing references, the papers by Eckstein, Shi, and Kevorkian[9,10] illustrate the application of the two-variable method to problems in celestial mechanics, in which, as is the present case, small perturbations are imposed upon a Keplerian motion.

The two time-like variables appropriate for this problem are

$$\phi = \theta \qquad \tilde{\phi} = \epsilon\theta \quad (6)$$

It is assumed that u and f may be expressed as explicit functions of ϕ and $\tilde{\phi}$ in the following asymptotic form

$$u = \sum_{i=0}^{N} u_i(\phi, \tilde{\phi})\epsilon^i + 0(\epsilon^{N+1}) \quad (7a)$$

$$f = \sum_{i=0}^{N} f_i(\phi, \tilde{\phi})\epsilon^i + 0(\epsilon^{N+1}) \quad (7b)$$

One can then calculate from Eq. (3) the following equations governing the first three terms in the expansions for u and f:

$$(\partial^2 u_0/\partial\phi^2) + u_0 = f_0^2 \quad (8a)$$

$$(\partial f_0/\partial\phi) = 0 \quad (8b)$$

$$(\partial^2 u_1/\partial\phi^2) + u_1 = -2(\partial^2 u_0/\partial\phi\partial\tilde{\phi}) + 2f_0f_1 - (f_0^2\cos\alpha/u_0^2) - f_0^2(\sin\alpha/u_0^3)(\partial u_0/\partial\phi) \quad (9a)$$

$$(\partial f_1/\partial\phi) = -(\partial f_0/\partial\tilde{\phi}) - (f_0^3/u_0^3)\sin\alpha \quad (9b)$$

$$(\partial^2 u_2/\partial\phi^2) + u_2 = -(\partial^2 u_0/\partial\tilde{\phi}^2) - 2(\partial^2 u_1/\partial\phi\partial\tilde{\phi}) + (f_1^2 + 2f_0f_2) + [(2f_0^2u_1/u_0^3) - (2f_0f_1/u_0^2)]\cos\alpha - (2f_0f_1/u_0^3)(\partial u_0/\partial\phi)\sin\alpha + (3u_1f_0^2\sin\alpha/u_0^4)(\partial u_0/\partial\phi) - (f_0^2\sin\alpha/u_0^3) \times [(\partial u_1/\partial\phi) + (\partial u_0/\partial\tilde{\phi})] \quad (10a)$$

$$(\partial f_2/\partial\phi) = -(\partial f_1/\partial\tilde{\phi}) - (3f_0^2f_1/u_0^3)\sin\alpha + [(3f_0^3u_1)/u_0^4]\sin\alpha \quad (10b)$$

According to Eqs. (4), the following initial values for the u_i, f_i and their derivatives must be imposed:

$$f_0(0, 0) = f_{00} \quad (11a)$$

$$u_0(0, 0) = f_{00}^2[1 + e_0\cos\omega_0] = 1 \quad (11b)$$

$$[\partial u_0(0, 0)/\partial\phi] = f_{00}^2 e_0 \sin\omega_0 \quad (11c)$$

and

$$f_i(0, 0) = 0 \quad (12a)$$

$$u_i(0, 0) = 0 \quad (12b)$$

$$[\partial u_i(0, 0)/\partial\phi] + [\partial u_{i-1}(0, 0)/\partial\tilde{\phi}] = 0 \quad (12c)$$

for $i \neq 0$.

4. Solutions Including First-Order Perturbations

The solution of Eq. (8) for u_0 and f_0 defines these variables in the form

$$f_0 = f_0(\tilde{\phi}) \quad (13a)$$

$$u_0 = f_0^2\{1 + e(\tilde{\phi})\cos[\phi - \omega(\tilde{\phi})]\} \quad (13b)$$

associated with Keplerian motion, where now the elements e and ω§ and the angular momentum f_0^{-1} are functions of $\tilde{\phi}$ and hence vary slowly with time.

§ Note that e and ω, although closely related to the usual elliptic parameters, are not osculating elements as generally used in astronomy, because they do not contain any short-period terms.

Comparison of Eq. (13) with Eq. (11) shows that the initial values of e, ω, and f_0 are the constants e_0, ω_0, and f_{00}, respectively, which were introduced in Eqs. (4). The unknown functions f_0, e, and ω are determined by the requirement that the expansion (7) be consistent and hence contain no terms proportional to ϕ to $0(\epsilon)$. Thus, the straightforward integration of Eqs. (9) provides the conditions governing the three unknown functions occurring in Eqs. (13), as well as the formal structure of the perturbation terms f_1 and u_1. The results are summarized below.

In order to eliminate an inconsistent term proportional to ϕ appearing in f_1, one must set

$$(df_0/d\tilde{\phi}) = -(\sin\alpha/2f_0{}^3)[(2 + e^2)/(1 - e^2)^{5/2}] \quad (14)$$

and the removal of mixed-secular terms proportional to $\phi \sin\phi$ and $\phi \cos\phi$ in u_1 [which are also inconsistent with the uniformity of the development in Eqs. (7)] leads to the conditions

$$d\omega/d\tilde{\phi} = \cos\alpha/f_0{}^4(1 - e^2)^{3/2} \quad (15)$$

$$de/d\tilde{\phi} = -\tfrac{3}{2}[e \sin\alpha/f_0{}^4(1 - e^2)^{3/2}] \quad (16)$$

The perturbation terms f_1 and u_1 can then be expressed in the form

$$f_1 = -\frac{\sin\alpha}{f_0{}^3} \sum_{k=1}^{\infty} \frac{b_{3k}}{k} \sin k(\phi - \omega) + f_{10}(\tilde{\phi}) \quad (17)$$

$$u_1 = -\frac{\cos\alpha}{f_0{}^2}\left[\frac{b_{20}}{2} + \sum_{k=2}^{\infty} \frac{b_{2k}}{1 - k^2} \cos k(\phi - \omega)\right] - \frac{\sin\alpha}{f_0{}^2} \sum_{k=2}^{\infty} \frac{kb_{2k}}{1 - k^2} \sin k(\phi - \omega) - 2\frac{\sin\alpha}{f_0{}^2} \sum_{k=2}^{\infty} \frac{b_{3k}}{k(1 - k^2)} \times \sin k(\phi - \omega) + 2f_0(\tilde{\phi}) f_{10}(\tilde{\phi}) + A(\tilde{\phi}) \sin(\phi - \omega) + B(\tilde{\phi}) \cos(\phi - \omega) \quad (18)$$

where the infinite series occurring in Eqs. (17) and (18) arise from the development of $u_0{}^{-n}$ in Eqs. (9) in its Fourier series¶ according to the formulas shown in Appendix A. The terms f_{10}, A, and B are unknown functions of $\tilde{\phi}$ arising in the solution of the homogeneous equations, and will be determined from conditions upon the terms of $0(\epsilon^2)$ of the solution [Eqs. (30–32)]. Equations (14–16) simplify considerably if

$$e_0 = 0 \quad (19)$$

since, in this case, the solution for the eccentricity is the trivial one

$$e(\tilde{\phi}) = 0 \quad (20)$$

This special case which corresponds to the solution of Tsien,[1] Ting and Brofman[2], and Nayfeh[3] is contained in the results of the present paper [Eqs. (27–29)]. Clearly, a solution for $e_0 = 0$ is of little practical value unless the stability of such a limiting case is also demonstrated. For a closely related problem, Zee[4] has given a solution to first order in the eccentricity by the stroboscopic method. Cohen[5] tackles the problem of small tangential thrust by assuming at the outset that the cosine of the flight path angle is equal to unity in the equations of motion, and claims that his results are more accurate than those of Zee[4] and remain valid as long as the eccentricity is less than 0.2.

Clearly, it is inconsistent to impose a priori bounds on the eccentricity, with no further justification. An important motivation for the present paper is the need for a thorough discussion of the foregoing assumption according to the general results contained in the solutions of Eq. (14–16).

The following three integrals are easily derived from Eqs. (14–16), and define the solution of this system completely

¶ These Fourier developments were also used in the earlier work of Refs. 9 and 10.

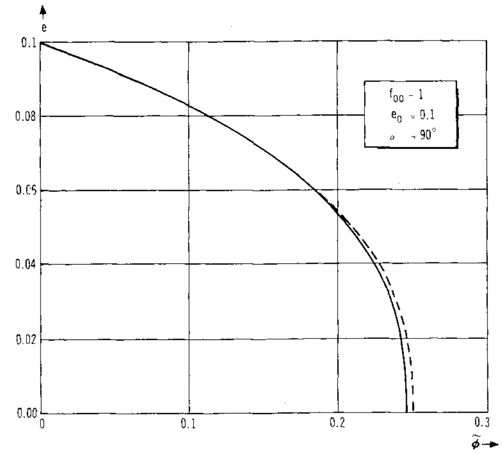

Fig. 2 Variation of eccentricity of an ascending satellite.

for all eccentricities††

$$\omega - \omega_0 = -\tfrac{2}{3} \cot\alpha \log(e/e_0) \quad (21)$$

$$[e^{2/3}/(1 - e^2)^{1/2}] = [e_0{}^{2/3}/(1 - e_0{}^2)^{1/2}](f_0/f_{00}) \quad (22)$$

and

$$-e^{2/3}(1 - e^2)^{1/2} + 3^{-1/4} F(\psi, k) = -e_0{}^{2/3}(1 - e_0{}^2)^{1/2} + 3^{-1/4} F(\psi_0, k) - \tfrac{15}{4}[\sin\alpha/(1 - e_0{}^2)^2](e_0{}^{8/3}/f_{00}{}^4)\tilde{\phi} \quad (23)$$

In Eq. (23), $F(\psi, k)$ is the elliptic integral of the first kind with amplitude ψ and modulus k, where

$$\cos\psi = [e^{2/3} - 1 + (3)^{1/2}]/[e^{2/3} - 1 - (3)^{1/2}] \quad (24a)$$

and

$$k^2 = \{[2 + (3)^{1/2}]/4\} = \sin^2(5\pi/12) \quad (24b)$$

Equation (23) gives $\tilde{\phi}$ as a function of e which can be inverted in principle and then used in Eqs. (21) and (22) to express ω and f as functions of $\tilde{\phi}$. The relationship between e and $\tilde{\phi}$ according to Eq. (23) is shown in Figs. 2-5. In Fig. 2, the initial conditions are $f_{00} = 1$, $e_0 = 0.1$ for an ascending orbit ($\alpha = 90°$). The dotted line is obtained from the approximate formula (27) for small e. It can be seen that e approaches zero rapidly as $\tilde{\phi}$ reaches the point where the solution has its singularity. According to Fig. 2, this occurs at $\tilde{\phi} = 0.24$, where $e = f_0 = 0$, saying that the angular momentum goes to infinity. For instance, if the initial thrust-to-weight ratio $\epsilon = 10^{-4}$, the singularity is at

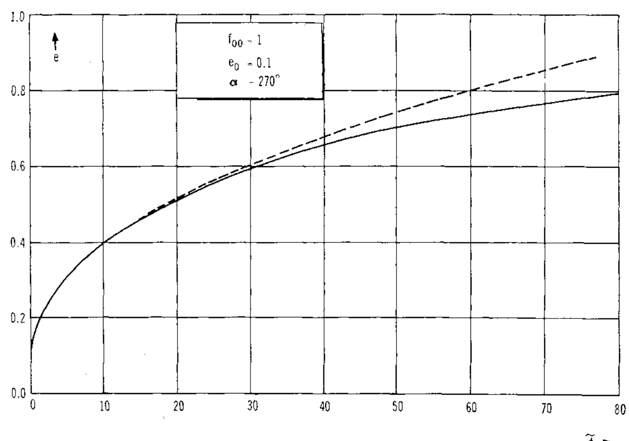

Fig. 3 Variation of eccentricity of a descending satellite.

†† Note when $e_0 \to 0$, the argument of pericenter ω is undefined as expected.

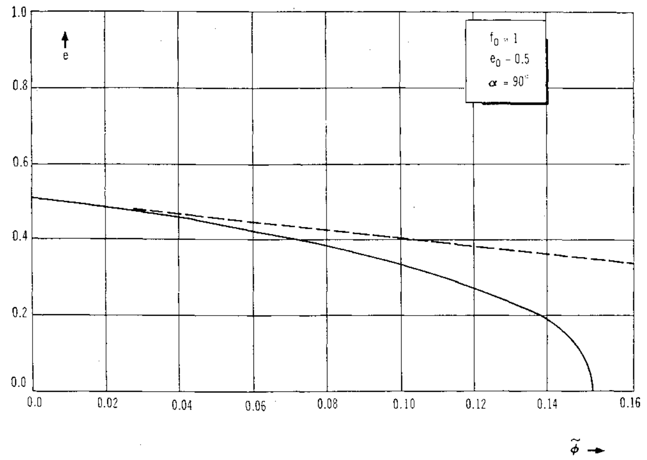

Fig. 4 Variation of eccentricity of an ascending satellite.

$\phi = 0.24 \times 10^4$ or after approximately 38 revolutions. Note that the approximate formula gives very good results in this case where e is small.

Figure 3 shows the corresponding plot for a descending orbit ($\alpha = 270°$) with the same initial conditions. Starting from a small value, the eccentricity grows very large for large $\tilde{\phi}$. Figures 4 and 5 are analogous to Figs. 2 and 3, but with $e_0 = 0.5$ as initial value. In this case, the singularity according to the solid curve in Fig. 4 is at $\tilde{\phi} = 0.015$, corresponding to about 24 revolutions if $\epsilon = 10^{-4}$. As expected for an initial eccentricity that large, the approximate formula (represented by the dotted line) is no longer a good approximation.

For Keplerian motion, the angular momentum is related to the elements e and a (semimajor axis).

$$f_0^{-2} = a(1 - e^2) \tag{25}$$

Use of this relation reduces Eq. (22) to the following simple and instructive form

$$e^4 a^3 = \text{const} = e_0^4 a_0^3 \tag{26}$$

Several interesting conclusions now follow from Eq. (26). For ascent from a satellite orbit, i.e., for $\sin\alpha > 0$, Eq. (23) shows, as expected, that the eccentricity must decrease according to Eq. (26). Thus, for ascending orbits, the eccentricity is a maximum initially and tends to zero as the orbit evolves. Therefore, the solution corresponding to $e_0 = 0$ is a stable one, and it is consistent to neglect higher order terms in the eccentricity if e_0 is small and $\sin\alpha > 0$.

Conversely, for descending orbits by means of constant small thrust, i.e., for $\sin\alpha > 0$, the semimajor axis decreases and the eccentricity approaches unity. Of course, in an actual problem involving an attracting body of finite dimensions, the particle will have collided with the surface before the eccentricity becomes unity.

The instability of the eccentricity for descending orbits precludes meaningful approximations of the solution wherein e is assumed small or constant. For example, the problem of descent to the lunar surface by small thrust from a parking orbit around the moon (cf. King[6]) cannot be treated by a small eccentricity theory unless the altitude of the parking orbit is also very small. However, under the assumption $\sin\alpha > 0$, $e_0 \ll 1$, Eqs. (21) and (23) give the following explicit results valid to first order in e

$$e = e_0\{1 - [(4 \sin\alpha)/f_{00}^4]\, \tilde{\phi}\}^{3/8} \tag{27}$$

$$f_0 = f_{00}\{1 - [(4 \sin\alpha)/f_{00}^4]\tilde{\phi}\}^{1/8} \tag{28}$$

and

$$\omega = \omega_0 - \tfrac{1}{4} \cot\alpha \log\{1 - [(4 \sin\alpha)/f_{00}^4]\tilde{\phi}\} \tag{29}$$

which are exactly the same as those of Nayfeh[3] when $e_0 \to 0$.

5. Higher Order Solutions

Equations (13), together with the relations for the slowly varying elements given by Eqs. (21–23) (or Eqs. (27–29) if $e_0 \ll 1$, $\sin\alpha > 0$), define an ellipse and the variation of its elements caused by the effect of the first-order perturbations. The evaluation of the unknown functions appearing in Eqs. (17) and (18) will define the orbit to a higher approximation. This requires the consideration of the second-order perturbations as represented by Eqs. (10).

The requirement that u_2 and f_2 as obtained from Eqs. (10) be bounded provides the following equations governing A, B, and f_{10};

$$\begin{aligned} df_{10}/d\tilde{\phi} = {} & [3(2 + 7e^2 + e^4)/2(1 - e^2)^{7/2}](\sin\alpha/f_0^4)f_{10} - \\ & [3e(4 + e^2)/2(1 - e^2)^{7/2}](\sin\alpha/f_0^5)B + Q_1(f_0, e) \end{aligned} \tag{30}$$

$$\begin{aligned} dB/d\tilde{\phi} = {} & [e(14 + 17e^2 - e^4)/(1 - e^2)^{7/2}](\sin\alpha/f_0^3)f_{10} - \\ & \tfrac{1}{2}[(7 + 25e^2 - 2e^4)/(1 - e^2)^{7/2}](\sin\alpha/f_0^4)B + Q_2(f_0, e) \end{aligned} \tag{31}$$

$$\begin{aligned} dA/d\tilde{\phi} = {} & -[2e(2 + e^2) \cos\alpha/f_0^3(1 - e^2)^{5/2}]\, f_{10} + \\ & [3e^2 \cos\alpha/f_0^4(1 - e^2)^{5/2}]B - (\sin\alpha/2f_0^4) \times \\ & [(7 - e^2)/(1 - e^2)^{5/2}]A + Q_3(f_0, e) \end{aligned} \tag{32}$$

where

$$Q_1 = -\frac{3}{2}\,\frac{\sin\alpha \cos\alpha}{f_0^7}\left[\sum_{k=2}^{\infty} \frac{b_{2k}\, b_{4k}}{1 - k^2} + \frac{1}{2}\, b_{40}\, b_{20}\right] \tag{33}$$

and Q_2 and Q_3 are given in Appendix B.

Since f_0 is a known function of e [Eqs. (16) and (22)], Eqs. (30) and (31) can be written as

$$\frac{df_{10}}{de} = -\frac{2 + 7e^2 + e^4}{(1 - e^2)^2}\, f_{10} + \frac{4 + e^2}{(1 - e^2)^2}\,\frac{B}{f_0} + g_1(e) \tag{34}$$

$$\frac{dB}{de} = -\frac{2}{3}\,\frac{14 + 17e^2 - e^4}{(1 - e^2)^2}\, f_0 f_{10} + \frac{1}{3}\,\frac{7 + 25e^2 - 2e^4}{e(1 - e^2)^2}\, B + g_2(e) \tag{35}$$

where

$$g_1(e) = Q_1[-\tfrac{3}{2}e \sin\alpha/f_0^4(1 - e^2)^{3/2}]^{-1} \tag{36}$$

and

$$g_2(e) = Q_2\{-\tfrac{3}{2}e \sin\alpha/f_0^4(1 - e^2)^{3/2}\}^{-1} \tag{37}$$

A set of solutions of the homogeneous equations corresponding to Eqs. (34) and (35) is obtained by complicated manipulations, the details of which are not indicated here. The

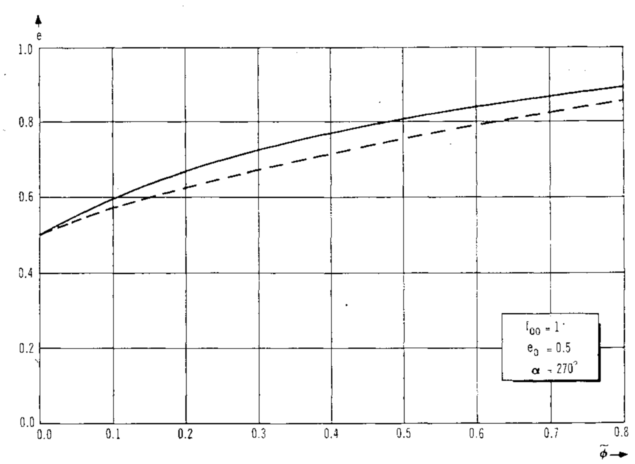

Fig. 5 Variation of eccentricity of a descending satellite.

following results may easily be verified by differentiation:

$$(f_{10})_{h_1} = [(2 + e^2)/3(1 - e^2)^{5/2} f_0^3] = V_{11} \quad (38)$$

$$(B)_{h_1} = (7 - e^2)e/3(1 - e^2)^{5/2} f_0^2 = V_{21} \quad (39)$$

The second set of solutions of the homogeneous equations (c.f. Martin and Reissner[16] and Morse and Feshbach[17]) is then obtained as

$$(f_{10})_{h_2} = V_{12} = \frac{1}{f_0^3} \int \frac{(4 + e^2)e}{(1 - e^2)^{7/2}} \int \frac{e^{5/3}}{(1 - e^2)^{1/2}} de\, de \quad (40)$$

$$(B)_{h_2} = V_{22} = \frac{e}{f_0^2(1 - e^2)^{3/2}} \int \frac{e^{5/3}}{(1 - e^2)^{1/2}} de + 2eV_{12} \quad (41)$$

The complete solution of Eqs. (34) and (35) is easily obtained by use of the method of variation of parameters[16,17]:

$$f_{10} = \left[C_1 + \int_{e_0}^{e} \frac{(g_1 V_{22} - g_2 V_{12})}{(V_{22} V_{11} - V_{12} V_{21})} de\right] V_{11} + \left[C_2 + \int_{e_0}^{e} \frac{(g_2 V_{11} - g_1 V_{21})}{(V_{22} V_{11} - V_{12} V_{21})} de\right] V_{12} \quad (42)$$

and

$$B = \left[C_1 + \int_{e_0}^{e} \frac{(g_1 V_{22} - g_2 V_{12})}{(V_{22} V_{11} - V_{12} V_{21})} de\right] V_{21} + \left[C_2 + \int_{e_0}^{e} \frac{(g_2 V_{11} - g_1 V_{21})}{(V_{22} V_{11} - V_{12} V_{21})} de\right] V_{22} \quad (43)$$

The integration constants can be expressed in terms of the initial values of A and B as

$$C_1 = \frac{f_{10}^{(0)} V_{22}^{(0)} - B^{(0)} V_{12}^{(0)}}{V_{22}^{(0)} V_{11}^{(0)} - V_{12}^{(0)} V_{21}^{(0)}} \quad (44)$$

$$C_2 = \frac{B^{(0)} V_{11}^{(0)} - f_{10}^{(0)} V_{21}^{(0)}}{V_{22}^{(0)} V_{11}^{(0)} - V_{12}^{(0)} V_{21}^{(0)}} \quad (45)$$

The superscript indicates the initial value of the corresponding variables. The values of $B^{(0)}$ and $f_{10}^{(0)}$ are given in Appendix C.

After substituting Eqs. (42) and (43) into Eq. (32) and using Eqs. (16) and (22) one obtains

$$dA/de = [(7 - e^2)/3\, e(1 - e^2)]A + g_3\,(e) \quad (46)$$

where $g_3(e)$ is given in Appendix B. The solution for A is expressed in terms of the initial value $A^{(0)}$ as

$$A = \frac{e^{7/2}(1 - e_0^2)}{e_0^{7/2}(1 - e^2)} A^{(0)} + \frac{e^{7/2}}{(1 - e^2)} \int_{e_0}^{e} g_3(e) e^{-7/2}(1 - e^2)\, de \quad (47)$$

The value of $A^{(0)}$ is given in Appendix C.

6. Solutions for Small Eccentricity

For small eccentricities, the previous results are simplified considerably. If the terms proportional to e^2 are neglected throughout Eqs. (30) and (31) one obtains

$$(df_{10}/d\tilde{\phi}) = (3 \sin\alpha/f_0^4)f_{10} - (6e \sin\alpha/f_0^5)B - (3 \cos\alpha \sin\alpha/f_0^7) \quad (48)$$

$$(dB/d\tilde{\phi}) = (14e \sin\alpha/f_0^3)f_{10} - (7 \sin\alpha/2f_0^4)B - (23e \cos\alpha \sin\alpha/2f_0^6) \quad (49)$$

As before, it is convenient to introduce f_0 as the independent variable by use of Eqs. (22) and (16). Equations (48) and (49) then transform to

$$(df_{10}/df_0) = -(3f_{10}/f_0) + (6e_0/f_{00}^{3/2}) \times (B/f_0^{1/2}) + (3 \cos\alpha/f_0^4) \quad (50)$$

$$\frac{dB}{df_0} = -14 \frac{e_0}{f_{00}^{3/2}} f_0^{3/2} f_{10} + \frac{7B}{2f_0} + \frac{23}{2} \frac{e_0 \cos\alpha}{f_{00}^{3/2} f_0^{3/2}} \quad (51)$$

The homogeneous solutions of the preceding equations are found to be

$$(f_{10})_{h_1} = (2/3f_0^3) \qquad (B)_{h_1} = (7e_0/3f_{00}^{3/2} f_0^{1/2}) \quad (52)$$

$$(f_{10})_{h_2} = \frac{9}{28} \frac{e_0^{14/3}}{f_{00}^7} f_0^4 \qquad (B)_{h_2} = \frac{3}{8} \frac{e_0^{11/3}}{f_{00}^{11/3}} f_0^{7/2} \quad (53)$$

It can easily be seen that the above homogeneous solutions are the leading terms of Eqs. (38–41). The complete solutions expressed in terms of the initial conditions become

$$f_{10} = (3 \cos\alpha/f_0^3) \log(f_0/f_{00}) + (3e_0 \sin\alpha \sin\omega_0/f_0^3) + (6e_0/7f_0^3)[(f_0^7/f_{00}^7) - 1](\cos\alpha \cos\omega_0 + 2 \sin\alpha \sin\omega_0) \quad (54)$$

$$B(\tilde{\phi}) = (f_0^{7/2}/f_{00}^{11/2})[\cos\alpha \cos\omega_0 + 2 \sin\alpha \sin\omega_0 + e_0(\cos\alpha \times \sin^2\omega_0 - \tfrac{5}{2} \sin\alpha \sin\omega_0 \cos\omega_0)] + \tfrac{21}{2} (e_0 \cos\alpha/f_{00}^{3/2}f_0^{1/2}) \log(f_0/f_{00}) - (e_0 \cos\alpha/4f_{00}^{3/2}f_0^{1/2})[1 - (f_0^4/f_{00}^4)] \quad (55)$$

Similarly Eq. (32) reduces to

$$(dA/d\tilde{\phi}) = -(4e \cos\alpha/f_0^3)f_{10} - \tfrac{7}{2}(\sin\alpha/f_0^4)A - \tfrac{63}{8}(e \sin^2\alpha/f_0^6) + (7e \cos^2\alpha/2f_0^6) \quad (56)$$

which has the following solution satisfying the initial conditions:

$$A(\tilde{\phi}) = [f_0^{7/2}/(f_{00})^{11/2}][-\cos\alpha \sin\omega_0 + 2 \sin\alpha \cos\omega_0 + e_0(6 - \tfrac{5}{2} \cos^2\omega_0) \sin\alpha + \tfrac{1}{2}e_0 \sin 2\omega_0 \cos\alpha] + \{[e_0(4 \cos\alpha \cot\alpha + 63 \sin\alpha)]/32f_{00}^{3/2}f_0^{1/2}\} \times [1 - (f_0^4/f_{00}^4)] - (3e_0/f_{00}^{3/2}f_0^{1/2}) \cos\alpha \cot\alpha \log(f_0/f_{00}) \quad (57)$$

where f_0 was given in Eq. (28).

7. Special Case of Circular Initial Orbit

For the limiting case $e_0 = 0$, the foregoing results reduce to the solution for circular orbits as discussed in Refs. 2 and 3. Substituting $e_0 = 0$ and $f_{00} = 1$ in the preceding solutions gives the results summarized below:

$$f = f_0 + \epsilon f_1 + 0(\epsilon^2) = f_0 + 3\epsilon(\cos\alpha/f_0^3) \log f_0 + 0(\epsilon^2) \quad (58)$$

$$u = u_0 + \epsilon u_1 + 0(\epsilon^2) = f_0^2 + \epsilon(1/f_0^2) \{\cos\alpha(6 \log f_0 - 1) + (f_0)^{11/2} \cos\alpha \cos[\phi + \cot\alpha \log f_0] + (f_0)^{11/2} \sin\alpha \sin[\phi + \cot\alpha \log f_0]\} + 0(\epsilon^2) \quad (59)$$

where

$$f_0 = (1 - 4\tilde{\phi}\sin\alpha)^{1/4} \quad (60)$$

which are in exact agreement with the solution obtained by Nayfeh[3] in Eqs. (3.23) and (3.25) of his paper.

8. Conclusions

In summary, the solutions for arbitrary eccentricity are given by Eqs. (13, 17, 18, 21–23, 42, 43, and 47). For small eccentricity, the results are expressed in simpler form by Eqs. (27, 29, 54, 55, and 57); for circular orbits these results reduce to Eqs. (58–60).

The present solution suffers from the same fundamental deficiency that is inherent in all the quoted approximate solutions for ascending orbits, viz., the results are only uniformly valid for the time interval measured from the initial instant of takeoff, $\phi = 0$ to $\phi = 0(1/\epsilon)$ subjected to the upper limit set by the assumption of small thrust (cf. Ting and Brofman[2]). Superficially, this is apparent in the singularity that occurs in the semimajor axis at the finite value of ϕ

[Eq. (28)] given by

$$\phi = f_{00}{}^4/4\epsilon \sin\alpha \tag{61}$$

It is clear that for sufficiently large times (of the order ϵ^{-1}) the original perturbation assumption is no longer valid, since the particle moves into a region where gravity is less and less important compared to the constant magnitude of the thrust, no matter how small one makes the latter. Although it is possible to develop a perturbation procedure valid for very long times, i.e., one in which the thrust forces are dominant, the problem of matching the two expansions is not a straightforward one. In fact, one can show that these two expansions do not overlap in the sense of singular perturbation theory (cf. Kaplun and Lagerstrom[11] for general principles of matching), and that some knowledge of the exact solution is required in order to connect the early phases of an ascending orbit to the solution very far away from the force center. It may be noted that Breakwell and Rauch[18] recently established such a connection for a special case by numerical integration. Treatment of this question, which is beyond the scope of the present paper, will be reported later.[19]

Appendix A: Fourier Expansion of the Functions

$$\frac{1}{[1 + e\cos(\phi - \omega)]^n} = \frac{b_{n0}}{2} + \sum_{k=1}^{\infty} b_{nk} \cos k(\phi - \omega)$$

where, for $k = 0, 1, 2 \ldots$

$$b_{1k} = [2/(1 - e^2)^{1/2}]\{[(1 - e^2)^{1/2} - 1]/e\}^k$$

$$b_{2k} = [2/(1 - e^2)^{3/2}]\{[(1 - e^2)^{1/2} - 1]/e\}^k \times [1 + k(1 - e^2)^{1/2}]$$

$$b_{3k} = \frac{2}{(1 - e^2)^{5/2}} \left(\frac{(1 - e^2)^{1/2} - 1}{e}\right)^k \times \left[1 + \frac{e^2}{2} + \frac{3}{2}k(1 - e^2)^{1/2} + \frac{k^2}{2}(1 - e^2)\right]$$

$$b_{4k} = \frac{2}{(1 - e^2)^{7/2}} \left(\frac{(1 - e^2)^{1/2} - 1}{e}\right)^k \times \left[1 + \frac{3}{2}e^2 + \frac{11 + 4e^2}{6} k(1 - e^2)^{1/2} + k^2(1 - e^2) + \frac{k^3}{6}(1 - e^2)^{3/2}\right]$$

Appendix B

$$Q_2 = -\left[\frac{3}{2}\frac{e^3}{(1 - e^2)^4} + \frac{b_{11}}{(1 - e^2)^{3/2}} - 3b_{20}\, b_{41}\right] \times \frac{\sin\alpha \cos\alpha}{2f_0{}^6} - \frac{\sin\alpha\cos\alpha}{2f_0{}^6} \sum_{k=2}^{\infty} \left\{\frac{b_{2k}}{k^2 - 1}\left[\frac{4}{e} b_{3k} - \frac{k^2}{e} b_{2k} + (k + 2)b_{3,k-1} + b_{3,k+1} + 6(b_{4,k+1} + b_{4,k-1})\right] - \frac{b_{2,k-1}\, b_{3k}}{k}\right\}$$

$$Q_3 = \frac{3}{8}\frac{e\sin^2\alpha}{f_0{}^6(1 - e^2)^5}[21 + 31e^2 + 2e^4] + \frac{1}{8}\frac{\sin^2\alpha}{f_0{}^6}\frac{(7 - e^2)}{(1 - e^2)^{5/2}} b_{22} + \frac{\sin^2\alpha}{4f_0{}^6}\sum_{k=2}^{\infty}\left\{\frac{b_{3k}}{k(k - 1)} \times [8b_{3,k-1} - (k - 1)^2 b_{2,k-1}] - \frac{k}{k - 1} b_{3,k-1}\, b_{2k} - \frac{(4b_{3k} - k^2 b_{2k})(b_{2k} + 2b_{3k})}{2e(k^2 - 1)}\right\} + \frac{\cos^2\alpha}{2f_0{}^6}\left\{\frac{e(7 - e^2)}{(1 - e^2)^4} + \sum_{k=2}^{\infty}\frac{b_{2k}}{k^2 - 1}(b_{3,k-1} + b_{3,k+1})\right\}$$

$$g_3 = -\frac{2}{3}\frac{f_0{}^4(1 - e^2)^{3/2}}{e\sin\alpha} Q_3 - \frac{2e\cot\alpha}{1 - e^2} B + \frac{4}{3}\frac{(2 + e^2)\cot\alpha}{(1 - e^2)} f_{10}$$

Appendix C: Initial Values of f_{10}, A, and B at $\tilde{\phi} = 0$

$$f_{10}{}^{(0)} = -(\sin\alpha/f_{00}{}^3)\sum_{k=1}^{\infty}(b_{3k}{}^{(0)}/k)\sin k\omega_0$$

$$A^{(0)} = \frac{\sin\alpha}{f_{00}{}^2}\left[b_{30}{}^{(0)}\cos\omega_0 + \frac{\epsilon_0(19 - e_0{}^2)}{4(1 - e_0{}^2)^{5/2}} - \frac{e_0(5 + e_0{}^2)}{4(1 - e_0{}^2)^{5/2}}\cos 2\omega_0\right] + \frac{\sin\alpha}{4f_{00}{}^2}\sum_{k=2}^{\infty}\left\{\frac{4b_{3k}{}^{(0)} - k^2 b_{3k}{}^{(0)}}{k(k + 1)} \times \cos(k - 1)\omega_0 - \frac{4(2k - 1)b_{3k}{}^{(0)} + k^2 b_{2k}{}^{(0)}}{k(k - 1)}\cos(k + 1)\omega_0\right\} - \frac{\cos\alpha}{f_{00}{}^2(1 - e_0{}^2)^{3/2}}\left(\sin\omega_0 - \frac{e_0}{2}\sin 2\omega_0\right) - \frac{\cos\alpha}{2f_{00}{}^2}\sum_{k=2}^{\infty} b_{2k}{}^{(0)} \times \left[\frac{\sin(k + 1)\omega_0}{k + 1} + \frac{\sin(k - 1)\omega_0}{k - 1}\right]$$

$$B^{(0)} = \frac{\sin\alpha}{f_{00}{}^2}\left[b_{30}{}^{(0)}\sin\omega_0 - \frac{e_0(5 + e_0{}^2)}{4(1 - e_0{}^2)^{5/2}}\sin 2\omega_0\right] + \frac{\sin\alpha}{4f_{00}{}^2}\sum_{k=2}^{\infty}\left[\frac{4b_{3k}{}^{(0)} - k^2 b_{2k}{}^{(0)}}{k(k + 1)}\sin(k + 1)\omega_0 + \frac{4(2k - 1)b_{3k}{}^{(0)} + k^2 b_{2k}{}^{(0)}}{k(k - 1)}\sin(k - 1)\omega_0\right] + \frac{\cos\alpha}{2f_{00}{}^2} \times \left[b_{20}{}^{(0)}\cos\omega_0 + \frac{e_0}{(1 - e_0{}^2)^{3/2}}(1 - \cos 2\omega_0)\right] + \frac{\cos\alpha}{2f_{00}{}^2} \times \sum_{k=2}^{\infty} b_{2k}{}^{(0)}\left[\frac{\cos(k + 1)\omega_0}{k + 1} - \frac{\cos(k - 1)\omega_0}{k - 1}\right]$$

where $b_{nk}{}^{(0)}$ is the value of b_{nk} for $e = e_0$ or $\phi = 0$.

References

[1] Tsien, H. S., "Take-off from satellite orbit," Jet Propulsion **23**, 233–236 (1953).

[2] Ting, L. and Brofman, S., "On take-off from circular orbit by small thrust," Z. Angew. Math. Mech. **44**, 417–428 (1964).

[3] Nayfeh, A. H., "Take-off from a circular orbit by a small thrust," AIAA Paper 65-688 (September 1965); also AIAA *Progress in Astronautics and Aeronautics: Methods in Astrodynamics and Celestial Mechanics*, edited by R. L. Duncombe and V. G. Szebehely (Academic Press Inc., New York, 1966), Vol. 17, 139–157.

[4] Zee, C. H., "Low constant tangential thrust spiral trajectories," AIAA J. **1**, 1581–1583 (1963).

[5] Cohen, M. J., "Low-thrust spiral trajectory of a satellite of variable mass," AIAA J. **3**, 1946–1949 (1965).

[6] King, J. C., "A natural first approximation to low-thrust trajectories between satellite orbits of the earth and moon," AIAA J. **3**, 1432–1439 (1965).

[7] Cole, J. D. and Kevorkian, J., "Uniformly valid asymptotic approximations for certain nonlinear differential equations," *Proceedings of the International Symposium on Non-linear Differential Equations and Non-linear Mechanics* (Academic Press Inc., New York, 1963), pp. 113–120.

[8] Kevorkian, J., "The two-variable expansion procedure for the approximate solution of certain non-linear differential equations," *Lectures in Applied Mathematics* (American Mathematical Society, Providence, R. I., 1966), Vol. 5, Pt. 3, pp. 206–275.

[9] Eckstein, M. C., Shi, Y. Y., and Kevorkian, J., "Satellite motion for arbitrary eccentricity and inclination around the smaller primary in the restricted three-body problem," Astron. J. **71**, 248–264 (May 1966).

[10] Eckstein, M. C., Shi, Y. Y., and Kevorkian, J., "Satellite

motion for all inclinations around an oblate planet," *Proceedings of IAU Symposium, No. 25, Thessaloniki, Greece*, edited by G. Contopoulos (to be published).

[11] Kaplun, S. and Lagerstom, P. A., "Asymptotic expansions of Navier-Stokes solutions for small Reynolds number," J. Math. Mech. **6**, 585–593 (1957).

[12] Morrison, J. A., "Comparison of the modified method of averaging and the two variable expansion procedure," SIAM Rev. **8**, 66–87 (1966).

[13] Minorsky, N., *Non-linear Oscillation* (D. Van Nostrand Company Inc., Princeton, N. J., 1962).

[14] Johnson, D. B. and Stumpf, L. W., "Perturbation solutions for low thrust rocket trajectories," AIAA J. **3**, 1934–1936 (October 1965).

[15] Irving, J. H., "Low thrust flight," *Variable Exhaust Velocity in Gravitational Field, Space Technology*, edited by H. Seifert (John Wiley & Sons Inc., New York, 1959), pp. 10-01–10-54.

[16] Martin, W. T. and Reissner, E., *Elementary Differential Equations* (Addison Wesley Publishing Company Inc., Reading, Mass., 1961).

[17] Morse, P. M. and Feshbach, H., *Methods of Theoretical Physics, Part I* (McGraw-Hill Book Company Inc., New York, 1955).

[18] Breakwell, J. V. and Rauch, H. E., "Optimum guidance for a low thrust interplanetary vehicle," AIAA J. **4**, 693–704 (1966).

[19] Shi, Y. Y. and Eckstein, M. C., "An approximate solution for ascending low-thrust trajectories without singularity," AIAA J. (to be published).

DECEMBER 1966 AIAA JOURNAL VOL. 4, NO. 12

Alfvén Waves and Induction Drag on Long Cylindrical Satellites

C. K. Chu* and R. A. Gross†

Columbia University, New York, N. Y.

This paper studies the induction drag of long cylindrical satellites and the Alfvén waves emitted from them. The potential and current distribution in the satellite is determined from the plasma ion and electron fluxes and the photoelectron emission. The induction drag can be significant, and at high satellite altitudes it is the major source of continual energy loss. The Alfvén wave drag is identified as part of the induction drag and is not a separate effect. Some numerical results are presented.

I. Introduction

THE electromagnetic effects and forces acting upon a satellite or a vehicle moving in the ionosphere or a rarefied plasma have been extensively studied and reviewed.[1–4] These reviews give a complete list of prior studies of this subject. Recently, Drell, Foley, and Ruderman[5] have called attention to the possibility of significant drag effects upon a satellite which might arise from Alfvén waves radiated from satellites of large physical dimension. They analyze a large conductor moving through a rarefied plasma that contains a magnetic field. By analogy with transmission line theory they predict a pair of Alfvén wings which may radiate from a large conducting satellite and they speculate further upon the consequences and possible uses of this power. Drell, Foley, and Ruderman, concerned with the physical mechanism by which current can leave the satellite, suggest photo-ejected electrons resulting from solar radiation as the mechanism for electrical contact between the satellite and the surrounding plasma. Although they avoid many physical details about the satellite itself, there is some radar evidence suggestive of long-range ionospheric disturbances that may originate from Alfvén waves.[6] It is the desire to clarify and analyze critically the work of Drell, Foley, and Ruderman which motivated this present study.

The forces and torques acting upon a satellite are relatively well known. They can be summarized as follows.

1) Aerodynamic: Charged particles and neutrals strike and leave the satellite surface after exchanging momentum with it. This effect includes outgassing of surfaces, ablation, Auger neutralization of incoming ions, etc. Such aerodynamic forces produce both drag and torque on a satellite.

2) Coulomb: The satellite electrical potential is in general not equal to that of the surrounding plasma. The resulting electric field surrounds the satellite for a distance of several Debye lengths,

$$\lambda_D = (\epsilon_0 kT/n_e e^2)^{1/2}$$

Charged particles, which never touch the satellite physically, are deflected by its electric field resulting in both a drag and a possible torque.

3) Induction or Lorentz: If the satellite has an electrical current flowing through it, which is not confined to the volume of the satellite itself, the interaction of this current with the earth's magnetic field will result in a drag and a possible torque.

4) Eddy current: A conducting satellite's motion may induce currents resulting from induced voltage ($\bar{V} \times \bar{B}$). These currents confined within the satellite interact with the magnetic field and produce a torque. There is no net drag so long as the magnetic field is constant over the spatial dimension of the satellite; i.e. $|\nabla B| \ll B/l$.

5) Radiation: Photons carry with them momentum and when they are absorbed on the satellite they result in a radiation pressure.

All of these effects have been studied in one case or another. The study by Wood and Hohl[2] for the satellite Echo II is particularly complete. For satellites in the ionosphere, the major sources of drag result from the aerodynamic and Lorentz forces. The Lorentz force on long conducting satellites was studied initially by Beard and Johnson.[7] They compute the charged particle flux onto a long conducting satellite in which there is a motion-induced electric field, and they determine the resulting Lorentz drag. They do not, however, consider a photoelectric flux of electrons nor do they concern themselves with possible waves generated in the plasma medium. The fact that the photoelectric current is often significant is shown by the measurements of Bourdeau et al.[8] on Explorer VIII. On the other hand, the work of Drell, Foley, and Ruderman ignore the detailed particle

Presented as Preprint 66-478 at the AIAA 4th Aerospace Sciences Meeting, Los Angeles, Calif., June 27–29, 1966; revision received July 22, 1966. This work was supported under NASA Grant NsG-302-63 and Astrosystems International Inc.

* Associate Professor of Engineering Science, School of Engineering and Applied Science. Member AIAA.

† Professor of Engineering Science, School of Engineering and Applied Science. Fellow AIAA.